

On the Capacity of Relaying with Finite Blocklength

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Abstract

In this paper, the relaying performance is studied under the finite blocklength regime. The overall error probability of relaying is derived. Moreover, we investigate the Blocklength-Limited capacity (BL-capacity) of relaying. We prove that the BL-capacity of relaying is quasiconcave in the overall error probability. Therefore, the BL-capacity has a global maximum value which can be achieved by choosing an appropriate error probability. Through numerical investigations, we validate our analytical model and compare the performance of relaying under the finite blocklength regime versus the Shannon capacity regime.

Index Terms

Finite blocklength regime, decode-and-forward, relaying, blocklength-limited capacity.

I. INTRODUCTION

In wireless communications, relaying [1]–[3] is well known as an efficient way to mitigate wireless fading by exploiting spatial diversity. Specifically, two-hop decode-and-forward (DF) relaying protocols significantly improve the capacity and quality of service [4]–[7]. However, all the above studies of the advantages of relaying are under the ideal assumption of communicating arbitrarily reliably at Shannon’s channel capacity.

For communication with arbitrarily small decoding error probability, coding is assumed to be performed using a block with an infinite length. If the codeword is restricted to a reasonable size, i.e. to a finite

blocklength, the error probability of the communication becomes no longer negligible. Hence, in the finite blocklength regime it is essential to consider the error probability while investigating the communication performance. Taking the error probability into account, [8] identifies a tight bound of coding rate of a single-hop transmission system. The authors show that the performance loss due to a finite blocklength is considerable and becomes more significant when the blocklength is relatively short.

Our work is motivated by the above observation that a shorter blocklength leads to a significant loss in the finite blocklength regime. For a two-hop relaying system (with equal time division), the blocklength of direct transmission is twice as long as the blocklength in each hop of relaying. Hence, one could suspect that relaying pays off less in the finite blocklength regime compared to the Shannon capacity regime because of halving the blocklength. However, to the best of our knowledge, the performance of relaying in the finite blocklength regime has not been studied in detail so far.

In this work, the finite blocklength relaying performance is investigated analytically. We first derive the overall error probability of relaying with a finite blocklength. Subsequently, we investigate the Blocklength-Limited capacity (BL-capacity) of relaying, which represents the number of bits decoded correctly at the destination per channel use. We prove that for a given coding rate the BL-capacity is quasiconcave in the overall error probability of the relaying system. Then, through numerical results we validate our analytical model and evaluate the relaying performance under finite blocklength regime in comparison to the Shannon capacity. Most importantly (and surprisingly), we show that relaying is more efficient in the finite blocklength regime than in the Shannon capacity regime.

The rest of the paper is organized as follows. Section II describes the system model. In Section III, the overall error probability of relaying is derived. Based on this, the BL-capacity of relaying is studied. Section IV contains our simulation results. Finally, we conclude our work in Section V.

II. SYSTEM MODEL

We consider a simple relaying scenario with a source, a destination and a DF relay as schematically shown in Fig. 1. The links between the above transceivers are referred to as the direct link (from the source to the destination), the backhaul link (from the source to the relay) and the relaying link (from the

relay to the destination). In general, we assume the direct link to be much weaker than the backhaul link as well as the relaying link. The entire system operates in a slotted fashion where time is divided into frames of length m (symbols). The blocklength of the coding over the channel in each frame is as long as the frame length m . In order to transmit data from the source to the destination, first a broadcasting frame is employed, followed by a relaying frame. During the broadcasting frame, the source transmits data to the relay as well as the destination. If the relay decodes the data correctly, it forwards the data to the destination during the subsequent relaying frame.

We consider a real Gaussian channel model with static channel gains and denote the channels (scalars) of the direct link, backhaul link and relaying link by h_1 , h_2 and h_3 , respectively. Moreover, the corresponding noise vectors of these links are denoted by \mathbf{n}_1 , \mathbf{n}_2 and \mathbf{n}_3 , which are independent and identically distributed (i.i.d.) real Gaussian vectors: $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_m)$, $\mathbf{n} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, where \mathbf{I}_m denotes an $m \times m$ identity matrix. In addition, the transmit power at the relay and the source is denoted by p_{tx} . Hence, the received signals at the destination and the relay in a broadcasting frame are given by: $\mathbf{y}_1 = h_1 \mathbf{x} + \mathbf{n}_1$ and $\mathbf{y}_2 = h_2 \mathbf{x} + \mathbf{n}_2$. Next, if the data is decoded correctly and forwarded by the relay, the received signal at the destination in a relaying frame is given by $\mathbf{y}_3 = h_3 \mathbf{x} + \mathbf{n}_3$. The transmitted signal \mathbf{x} and received signals \mathbf{y}_1 , \mathbf{y}_2 and \mathbf{y}_3 are real m -dimensional vectors. We assume perfect channel state information (CSI) at the receivers and in particular at the source. In addition, the destination is assumed to apply maximum ratio combining based on the CSI where the combined channel gain is given by $h_1^2 + h_3^2$. Thus, the received signal-to-noise ratio (SNR) at the relay and the received SNR at the destination under maximum ratio combining are given by $\gamma_2 = h_2^2 p_{\text{tx}} / \sigma^2$ and $\gamma_{\text{MRC}} = (h_1^2 + h_3^2) p_{\text{tx}} / \sigma^2$.

Under the finite blocklength regime, decoding errors may occur. We assume that both the relay and the destination reliably detect the errors. Based on this protocol, the relay does not forward the block to the destination when an error occurs. In addition, if a decoding error occurs at the destination, the effectively transmitted information s (i.e. payload bits) of the two-frame relaying equals zero. On the contrary, if no error occurs, the effectively transmitted information equals $s = m \cdot r$, where r is the coding rate (in bits per channel use) employed over a block of m symbols in each hop/frame of relaying. Finally, the

BL-capacity C_{BL} of two-frame relaying is defined as the average effectively transmitted information per channel use, given by $C_{\text{BL}} = \text{E}[s]/2m$.

III. THE BL-CAPACITY OF RELAYING

In this section, the relaying behavior is investigated in the finite blocklength regime analytically. First, the overall error probability is analyzed. Based on this, we further study the BL-capacity of relaying.

A. The Overall Error Probability of Relaying

For the real additive white Gaussian noise (AWGN) channel, [8, Theorem 54] derives a tight bound for the coding rate of a single-hop transmission system. With blocklength m , error probability ε and SNR γ , the coding rate (in bits per channel use) is given by: $r = \frac{1}{2} \log_2(1 + \gamma) - \sqrt{\frac{1}{2m} \left(1 - \frac{1}{(1+\gamma)^2}\right)} Q^{-1}(\varepsilon) \log_2 e + \frac{O(\log_2 m)}{m}$, where $Q^{-1}(\cdot)$ is the inverse Q-function and as usual the Q-function is given by $Q(w) = \int_w^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Hence, the above result can be reformulated to a real Gaussian channel model with channel gain h^2 :

$$r = R(h^2, \varepsilon, m) \approx C(h^2) - \sqrt{\frac{1}{2m} (1 - 2^{-4C(h^2)})} Q^{-1}(\varepsilon) \log_2 e, \quad (1)$$

where $C(h^2)$ is the Shannon capacity function of a real channel with gain h^2 : $C(h^2) = \frac{1}{2} \log_2 \left(1 + \frac{h^2 p_{\text{tx}}}{\sigma^2}\right)$.

Hence, for a single frame transmission with coding rate r and blocklength m , if the transmitter has perfect CSI, the decoding error probability at the receiver is given by:

$$\varepsilon = P^e(h^2, r, m) = Q \left(\frac{C(h^2) - r}{\sqrt{\frac{1}{2m} (1 - 2^{-4C(h^2)})} \log_2 e} \right). \quad (2)$$

Obviously, $P^e(h^2, r, m)$ is a strictly increasing function of r as the Q-function is a strictly decreasing function.

In the assumed relaying system, as we consider maximum ratio combining at the receiver, the coding rate on different links needs to be the same. Hence, with coding rate r the decoding error probability at the relay is given by $\varepsilon_2 = P^e(h_2^2, r, m)$ while the decoding error probability at the destination under maximum ratio combining is given by $\varepsilon_{\text{MRC}} = P^e(h_1^2 + h_3^2, r, m)$.

The coding rate is determined by the source based on the CSI of all the links and in particular the bottleneck link of the system which is either the backhaul link or the combined link. Recall that we assume both of these links to be significantly stronger than the direct link: $\min\{h_2^2, h_1^2 + h_3^2\} \gg h_1^2$. As the coding rate of a block is chosen based on h_2^2 or $h_1^2 + h_3^2$, the probability of decoding the block correctly at the destination just relying on the much lower (direct link) channel gain h_1^2 is negligible, which means $P^e(h_1^2, r, m) \approx 1$. Therefore, the overall error probability ε_R of the two-frame relaying equals the probability of the intersection of the following two events: "an error occurs at the relay" and "an error occurs at the destination after maximum ratio combining". This leads to the following proposition:

Proposition 1. *If the bottleneck link (either the backhaul link or the combined link) is significantly stronger than the direct link, with coding rate r and blocklength m in each frame/hop, the overall error probability of relaying is given by:*

$$\begin{aligned} \varepsilon_R &= \varepsilon_2 + (1 - \varepsilon_2) \varepsilon_{\text{MRC}} \\ &= P^e(h_2^2, r, m) + [1 - P^e(h_2^2, r, m)] P^e(h_1^2 + h_3^2, r, m). \end{aligned} \quad (3)$$

Hence, we immediately have a lower bound for ε_R :

$$\varepsilon_R \geq \max\{\varepsilon_{\text{MRC}}, \varepsilon_2\}. \quad (4)$$

Equations (3) and (4) indicate that the overall error probability of relaying is directly subject to the error probabilities of the backhaul link and the combined link. Based on (2), the error probability of the two links are subject to the coding rate. As the coding rate is determined by the source based on the channel conditions of the bottleneck link, we further study an upper bound of the overall error probability by investigating the following two scenarios with different bottleneck links:

- **Case $h_2^2 \geq h_1^2 + h_3^2$:** The backhaul link is stronger than the combined link. Hence, the bottleneck link is the combined link. Therefore, $P^e(h_2^2, r, m) \leq P^e(h_1^2 + h_3^2, r, m)$. We have the following relationship between the overall error probability ε_R and the error probability of the combined link ε_{MRC} :

$$\varepsilon_R \leq 2\varepsilon_{\text{MRC}} - (\varepsilon_{\text{MRC}})^2. \quad (5)$$

When $h_2^2 \gg h_1^2 + h_3^2$, we have $\varepsilon_R \approx \varepsilon_{\text{MRC}}$.

- **Case** $h_2^2 < h_1^2 + h_3^2$: The bottleneck link is the backhaul link. Similarly, the upper bound of ε_R is:

$$\varepsilon_R \leq 2\varepsilon_2 - (\varepsilon_2)^2. \quad (6)$$

When $h_1^2 + h_3^2 \gg h_2^2$, we have $\varepsilon_R \approx \varepsilon_2$.

Combining (5) and (6) with (4), the overall error probability of relaying is bounded by:

$$2\varepsilon_\star - (\varepsilon_\star)^2 \geq \varepsilon_R \geq \varepsilon_\star, \quad (7)$$

where $\varepsilon_\star = \max\{\varepsilon_{\text{MRC}}, \varepsilon_2\}$ is the error probability of the bottleneck link. Hence, we have $\varepsilon_R \approx \varepsilon_\star$ if $h_1^2 + h_3^2 \gg h_2^2$ or $h_1^2 + h_3^2 \ll h_2^2$. On the other hand, if $h_1^2 + h_3^2 \approx h_2^2$, this results in $\varepsilon_{\text{MRC}} \approx \varepsilon_2$. Therefore, based on (3) we have $\varepsilon_R \approx 2\varepsilon_\star - (\varepsilon_\star)^2$ and in particular we have $\varepsilon_R \approx 2\varepsilon_\star$ with a low ε_\star , e.g., $\varepsilon_\star < 0.1$.

For given error probabilities of the backhaul link and the combined link, the coding rate can be determined by the source based on (1) and is given by:

$$r = R(\min\{h_2^2, h_2^2 + h_3^2\}, \max\{\varepsilon_2, \varepsilon_{\text{MRC}}\}, m). \quad (8)$$

According to (8) (with given channel gains and blocklength) r is only influenced by the error probability of the bottleneck link ε_\star . Moreover, (2) and (3) show that the overall error probability ε_R is fully determined by r . As a result, ε_R is totally determined by ε_\star . The relationship between the overall error probability ε_R and the bottleneck link error probability ε_\star is given in the following proposition:

Proposition 2. *In a two-hop relaying network where the bottleneck link is significantly stronger than the direct link, for given channel gains and blocklength, the overall error probability ε_R is strictly increasing in the error probability of the bottleneck link ε_\star .*

Proof: See Appendix A. ■

B. The BL-Capacity of Relaying

Based on (2), ε_\star goes to 1 as long as the coding rate r tends to infinity. At the same time, ε_R also tends to 1, which makes the destination decode nothing correctly. Therefore, it is inappropriate to measure the relaying performance only by the coding rate. Combining the coding rate with the error probability, we

study the BL-capacity of relaying C_{BL} which is the average of the effectively transmitted information rate between the source and the destination. Hence, (as C_{BL} is the average over the random variable s which is defined in Section II as the effectively transmitted information per slot) C_{BL} is given by:

$$C_{BL} = \frac{E[s]}{2m} = \frac{(1 - \varepsilon_R) mr}{2m} = \frac{(1 - \varepsilon_R) r}{2}. \quad (9)$$

We then have the following proposition:

Proposition 3. *In a relaying system where the bottleneck link is significantly stronger than the direct link, for given channel gains and blocklength, if $\varepsilon_R < 0.5$, the BL-capacity C_{BL} is concave in the coding rate r .*

Proof: See Appendix B. ■

Note that according to Equation (1) the coding rate r is strictly increasing in the error probability of the bottleneck link ε_* , and that according to Proposition 2 ε_R is strictly increasing in ε_* . Hence, r is strictly increasing in ε_* or ε_R . Therefore, we have the following corollary of Proposition 3:

Proposition 4. *In a relaying system where the bottleneck link is significantly stronger than the direct link, for given channel gains, if $\varepsilon_R < 0.5$, the BL-capacity C_{BL} is quasiconcave in ε_R or ε_* .*

Proof: See Appendix C. ■

Proposition 4 shows that the BL-capacity of relaying has a global maximum which can be achieved by choosing a error probability of ε_R or ε_* .

C. Special Case: Multi-hop Relaying

Although in this paper we are interested in the relaying performance with maximum ratio combining, it should be mentioned that the above propositions can be simply extended to a multi-hop relaying scenario (without maximum ratio combining). A multi-hop relaying scenario differs from the assumed scenario of this work in that its bottleneck link is either the backhaul link or the relaying link. Hence, the error probability of the bottleneck link is given by $\varepsilon_* = \max\{\varepsilon_2, \varepsilon_3\}$, where ε_3 is the error probability of the relaying link. In addition, the overall error probability of the multi-hop relaying is given by $\varepsilon_R = \varepsilon_2 + (1 - \varepsilon_2) \varepsilon_3$. Based on the above expressions for ε_* and ε_R , Propositions 2, 3 and 4 hold in the multi-hop relaying scenario as well.

IV. NUMERICAL RESULTS AND DISCUSSION

In this section, we present some numerical results obtained by simulations. We show the relationship between the overall error probability and the error probability of the bottleneck link. In addition, we evaluate the relaying performance with a finite blocklength in comparison to the Shannon capacity. In the simulation, we consider an outdoor urban scenario and the distances of the backhaul, relaying and direct links are set to 200 m, 200 m and 360 m (we vary the system topology only in Fig. 2). We set the transmit power p_{tx} equal to 22 dBm and noise power to -95 dBm, respectively. In addition, we utilize the well-known COST231 [9] model for calculating the path-loss while the center frequency is set to equal 2 GHz. As we consider static channels, in the simulation the channel gains are assumed to be fully subject to the path-loss. Based on the above topology setting, we have $h_2^2 = h_3^2 \gg h_1^2$ (in the following simulations except Fig. 2).

The relationship between the overall error probability and the error probability of the bottleneck link is shown in Fig. 2. It is shown that ε_{R} is increasing in ε_{\star} , which matches Proposition 2. Moreover, it matches our analysis that ε_{R} is approximately linearly increasing with ε_{\star} , i.e., $\varepsilon_{\text{R}} \approx k\varepsilon_{\star}$. More specifically, $k = 2$ when $h_1^2 + h_3^2 = h_2^2$ which means both of the two links equally limit the system performance. In addition, $k = 1$ if the gap between the channel gains of the backhaul link and the combined link is considerable, e.g., one is 20% higher than the other one. Finally, $1 < k < 2$ when the gap between the channel gains of the two links is neither negligible nor significantly high.

The comparison between the BL-capacity and the Shannon capacity of relaying is shown in Fig. 3. We plot the corresponding capacities of (non-relay) direct transmission as references. Firstly, as stated by Proposition 4, we observe from the plot that the BL-capacity curves are quasiconcave in the overall error probability. At the same time, the Shannon capacities are constant over the error probability as the channels are static. In particular, the BL-capacities with a short blocklength are more sensitive to the overall error probability. Therefore, the optimization of the BL-capacity is more important for the short blocklength systems. Secondly, the maximal BL-capacities of relaying with short and long blocklengths are achieved by different overall error probabilities. Hence, the optimal solution of maximizing the BL-

capacity of relaying is also influenced by the blocklength. Thirdly, relaying is significantly superior to direct transmission regarding both metrics, the Shannon capacity as well as the BL-capacity. In other words, in comparison to direct transmission, relaying improves the performance in both the Shannon capacity regime and the finite blocklength regime. Although in Fig. 3. we consider a lower SNR scenario, it should be mentioned that this gain of relaying of course depends on the scenario (i.e. there are scenarios where the gain is larger while in other scenarios the gain is lower).

We finally consider the performance advantage of relaying in the finite blocklength regime by normalizing the capacities from Fig. 3 and show them in Fig. 4. The capacity ratios in Fig. 4 are not normalized by the same value. Instead, we normalize the relaying capacities by relaying Shannon capacity and normalize the direct transmission capacities by the direct transmission Shannon capacity, respectively. By doing so, we observe the performance advantages of either relaying or direct transmission in the finite blocklength regime in comparison to the Shannon capacity regime.

From this, we firstly observe in Fig. 4 that a finite blocklength introduces a performance loss (in comparison to the Shannon capacity) to both cases, relaying and direct transmission. Secondly and surprisingly, the performance loss due to a finite (relatively shorter) blocklength m in the relaying case is much smaller than expected, and the performance loss due to a finite (relatively longer) blocklength $2m$ in the direct transmission scheme is larger than that. Especially, with a low error probability, relaying halves the performance loss in comparison to direct transmission. In other words, based on the numerical results we observe a performance advantage of relaying in the finite blocklength regime: Relaying is more efficient in the finite blocklength regime than in the Shannon capacity regime in comparison to direct transmission. Moreover, comparing the two sub-figures we find that this performance advantage of relaying becomes more notable for short blocklengths. Hence, in the finite blocklength regime relaying becomes even more promising with short blocklengths.

The explanation¹ of the performance advantage of relaying in the finite blocklength regime is that relaying achieves a higher SNR at the receiver in each frame than the received SNR at the destination

¹A rigorous proof for the performance advantage of relaying in finite blocklength regime will be presented in our future work.

of direct transmission. Consider the scenario where the equivalent Shannon capacity of relaying and the Shannon capacity of direct transmission are the same, i.e., $\frac{1}{2} \min \{C(h_2^2), C(h_1^2 + h_3^2)\} = C(h_1^2)$. Then, we have $\frac{p_{\text{tx}} \cdot \min\{h_2^2, h_1^2 + h_3^2\}}{\sigma^2} > \frac{p_{\text{tx}} \cdot h_1^2}{\sigma^2}$. Hence, even if their Shannon capacities are the same, relaying is still able to improve the reliability of the transmission in each frame as the receiver in each frame has a higher received SNR. In other words, relaying reduces the error probability. Thus, regarding the BL-capacity, when relaying and direct transmission have the same (overall) error probability, relaying has a higher BL-capacity than direct transmission.

V. CONCLUSION

In this work, we investigated the relaying performance under the finite blocklength regime. The overall error probability as well as the BL-capacity of relaying are derived. We proved that the overall error probability is strictly increasing in the error probability of the bottleneck link. In addition, we proved that the BL-capacity is concave in the coding rate and quasiconcave in the overall error probability of relaying. Therefore, the BL-capacity of relaying has a global maximum which can be achieved by choosing an appropriate overall error probability. Then, through numerical results we validated our analysis and evaluated the relaying performance under the finite blocklength regime. More importantly, we found that relaying has a performance advantage in the finite blocklength regime in comparison to the Shannon capacity regime, especially for short blocklength systems.

APPENDIX

A. Proof of the Proposition 2

We provide here the proof under the situation $h_2^2 \geq h_1^2 + h_3^2$, the other case can be proved similarly.

$$\textit{Proof: } h_2^2 \geq h_1^2 + h_3^2,$$

\Rightarrow the coding rate is decided by the source based on the combined channel, giving: $r = R(h_1^2 + h_3^2, \varepsilon_{\text{MRC}}, m)$.

$$\Rightarrow \varepsilon_{\text{R}} = \text{P}^e[h_2^2, R(h_1^2 + h_3^2, \varepsilon_{\text{MRC}}, m), m] + (1 - \text{P}^e[h_2^2, R(h_1^2 + h_3^2, \varepsilon_{\text{MRC}}, m), m]) \cdot \varepsilon_{\text{MRC}}.$$

\Rightarrow the derivative of ε_R with respect to ε_{MRC} is given by:

$$\begin{aligned} \frac{\partial \varepsilon_R}{\partial \varepsilon_{\text{MRC}}} &= \frac{\partial \varepsilon_R}{\partial r} \frac{\partial r}{\partial \varepsilon_{\text{MRC}}} \\ &= \left[\frac{\partial \varepsilon_{\text{MRC}}}{\partial r} (1 - \varepsilon_2) + \frac{\partial \varepsilon_2}{\partial r} (1 - \varepsilon_{\text{MRC}}) \right] \frac{\partial r}{\partial \varepsilon_{\text{MRC}}}. \end{aligned} \quad (10)$$

$R(h_1^2 + h_3^2, \varepsilon_{\text{MRC}}, m)$ is the increasing in ε_{MRC} as well as $P^e(h^2, r, m)$ is increasing in r ,

$$\Rightarrow \frac{\partial \varepsilon_{\text{MRC}}}{\partial r} > 0, \frac{\partial \varepsilon_2}{\partial r} > 0 \text{ and } \frac{\partial r}{\partial \varepsilon_{\text{MRC}}} > 0.$$

$$0 \leq \varepsilon_2 \leq 1 \text{ and } 0 \leq \varepsilon_{\text{MRC}} \leq 1,$$

$$\Rightarrow \text{based on (10), } \frac{\partial \varepsilon_R}{\partial \varepsilon_{\text{MRC}}} > 0.$$

$$\Rightarrow \varepsilon_R \text{ is strictly increasing in } \varepsilon_{\text{MRC}}. \quad \blacksquare$$

B. Proof of the Proposition 3

Here we provide the proof under the situation that the bottleneck link is the backhaul link, the other situation can be analyzed similarly.

Proof: The first derivative of C_{BL} with respect to r_3 is: $\frac{\partial C_{\text{BL}}}{\partial r} = \frac{1 - \varepsilon_R}{2} - \frac{1}{2} \frac{\partial \varepsilon_R}{\partial r}$, where $\frac{\partial \varepsilon_R}{\partial r} = \frac{1 - \varepsilon_2}{2} \frac{\partial \varepsilon_{\text{MRC}}}{\partial r} + \frac{1 - \varepsilon_{\text{MRC}}}{2} \frac{\partial \varepsilon_2}{\partial r} > 0$.

\Rightarrow the second derivative of C_{BL} with respect to r_3 :

$$\begin{aligned} \frac{\partial^2 C_{\text{BL}}}{\partial^2 r} &= \frac{r}{2} \frac{\partial \varepsilon_{\text{MRC}}}{\partial r} \frac{\partial \varepsilon_2}{\partial r} - \frac{r(1 - \varepsilon_{\text{MRC}})}{4} \frac{\partial^2 \varepsilon_2}{\partial^2 r} \\ &\quad - \frac{r(1 - \varepsilon_2)}{4} \frac{\partial^2 \varepsilon_{\text{MRC}}}{\partial^2 r} - \frac{1 - \varepsilon_{\text{MRC}}}{2} \frac{\partial \varepsilon_2}{\partial r} \\ &\quad - \frac{1 - \varepsilon_2}{2} \frac{\partial \varepsilon_{\text{MRC}}}{\partial r}. \end{aligned} \quad (11)$$

\Rightarrow Proposition 3 holds if the right side of (11) is negative.

We further obtain the first and second derivative of ε_* with respect to the coding rate r based on (2):

$$\frac{\partial \varepsilon_*}{\partial r} = \frac{e^{-\frac{m(C(h_*^2) - r)}{(1 - 2^{-4C(h_*^2)})(\log_2 e)^2}}}{2\sqrt{\pi}(1 - 2^{-4C(h_*^2)})(\log_2 e)^2}, \quad (12)$$

$$\frac{\partial^2 \varepsilon_*}{\partial^2 r} = \frac{m(C(h_*^2) - r) e^{-\frac{m(C(h_*^2) - r)}{(1 - 2^{-4C(h_*^2)})(\log_2 e)^2}}}{\sqrt{\pi}(1 - 2^{-4C(h_*^2)})^2 (\log_2 e)^4}, \quad (13)$$

where $\varepsilon_* \in \{\varepsilon_{\text{MRC}}, \varepsilon_2\}$ and h_*^2 is the channel gain of the bottleneck link: $h_*^2 \in \{h_2^2, h_1^2 + h_3^2\}$.

Note that we consider a non-extreme error transmission where $\varepsilon_2 < 0.5$. Therefore, $\varepsilon_* \leq \varepsilon_2 < 0.5$. Based on (2), the coding rate of each frame r determined by the source based on the channel quality of bottleneck link is lower than the Shannon capacity of the link: $C(h_*^2) > r$.

\Rightarrow both $\frac{\partial \varepsilon_*}{\partial r}$ and $\frac{\partial^2 \varepsilon_*}{\partial^2 r}$ are positive.

The bottleneck link is the backhaul link

$\Rightarrow \forall r, \varepsilon_2 = P^e(h_2^2, r, m) > \varepsilon_{\text{MRC}} = P^e(h_1^2 + h_3^2, r, m)$ and $\frac{\partial \varepsilon_2}{\partial r} > \frac{\partial \varepsilon_{\text{MRC}}}{\partial r}$.

\Rightarrow based on (11), we have:

$$\begin{aligned} 4 \frac{\partial^2 C_{\text{BL}}}{\partial^2 r} &= 2r \frac{\partial \varepsilon_{\text{MRC}}}{\partial r} \frac{\partial \varepsilon_2}{\partial r} - (1 - \varepsilon_{\text{MRC}}) \left[r \frac{\partial^2 \varepsilon_2}{\partial^2 r} + 2 \frac{\partial \varepsilon_2}{\partial r} \right] \\ &\quad - (1 - \varepsilon_2) \left[r \frac{\partial^2 \varepsilon_{\text{MRC}}}{\partial^2 r} + 2 \frac{\partial \varepsilon_{\text{MRC}}}{\partial r} \right] \\ &< 2r \frac{\partial \varepsilon_{\text{MRC}}}{\partial r} \frac{\partial \varepsilon_2}{\partial r} - r (1 - \varepsilon_{\text{MRC}}) \frac{\partial^2 \varepsilon_2}{\partial^2 r} \\ &< 2r \left(\frac{\partial \varepsilon_2}{\partial r} \right)^2 - r (1 - \varepsilon_2) \frac{\partial^2 \varepsilon_2}{\partial^2 r}. \end{aligned} \tag{14}$$

$\Rightarrow \frac{\partial^2 C_{\text{BL}}}{\partial^2 r} < 0$ if $2 \left(\frac{\partial \varepsilon_2}{\partial r} \right)^2 / \frac{\partial^2 \varepsilon_2}{\partial^2 r} < 1 - \varepsilon_2$, where $1 - \varepsilon_2 \geq 1 - \varepsilon_{\text{R}} > 0.5$.

$2 \left(\frac{\partial \varepsilon_2}{\partial r} \right)^2 / \frac{\partial^2 \varepsilon_2}{\partial^2 r} = \frac{\exp\left\{-\frac{m(C(h_2^2)-r)^2}{(1-2^{-4}C(h_2^2))(\log_2 e)^2}\right\}}{2\sqrt{\pi}m(C(h_2^2)-r)} \ll 0.5$. Especially, for a relative big m or/and a big gap between $C(h_2^2)$ and r , $2\sqrt{\pi}m(C(h_2^2)-r) \gg 1$ while $\exp\left\{-\frac{m(C(h_2^2)-r)^2}{(1-2^{-2}C(h_2^2))(\log_2 e)^2}\right\} \ll 1$. For example, if $m = 100$ and $r = \frac{4}{5} C(h_2^2) = 0.4$ bit/ch.use, $2 \left(\frac{\partial \varepsilon_2}{\partial r} \right)^2 / \frac{\partial^2 \varepsilon_2}{\partial^2 r} \approx 8.2 \times 10^{-3} \ll 0.5$.

$\Rightarrow \frac{\partial^2 C_{\text{BL}}}{\partial^2 r} < 0$ for the non-extreme error transmissions. Hence, the BL-capacity C_{BL} is a concave function of the coding rate r . ■

C. Proof of the Proposition 4

Proof: r is strictly increasing in ε_{R} , $\varepsilon_{\text{R}} \in [0, 1]$.

$\Rightarrow \forall x < y, x, y \in [0, 1]$ and $\lambda \in [0, 1]$, we have $r|_{\varepsilon_2=x} < r|_{\varepsilon_2=\lambda x+(1-\lambda)y} < r|_{\varepsilon_2=y}$.

C_{BL} is concave in r ,

$\Rightarrow \min\left\{C_{\text{BL}}(r|_{\varepsilon_2=x}), C_{\text{BL}}(r|_{\varepsilon_2=y})\right\} \leq C_{\text{BL}}(r|_{\varepsilon_2=\lambda x+(1-\lambda)y})$.

$\Rightarrow C_{\text{BL}}$ is quasiconcave in ε_{R} , $\varepsilon_{\text{R}} \in [0, 1]$. And similarly, we can prove that C_{BL} is quasiconcave in ε_* . ■

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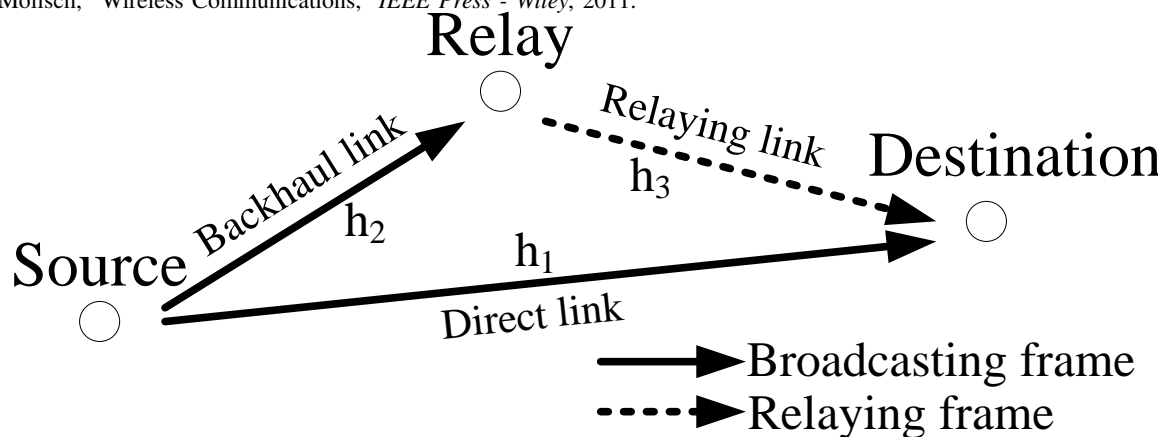


Fig. 1. Example of the considered single relay system scenario.

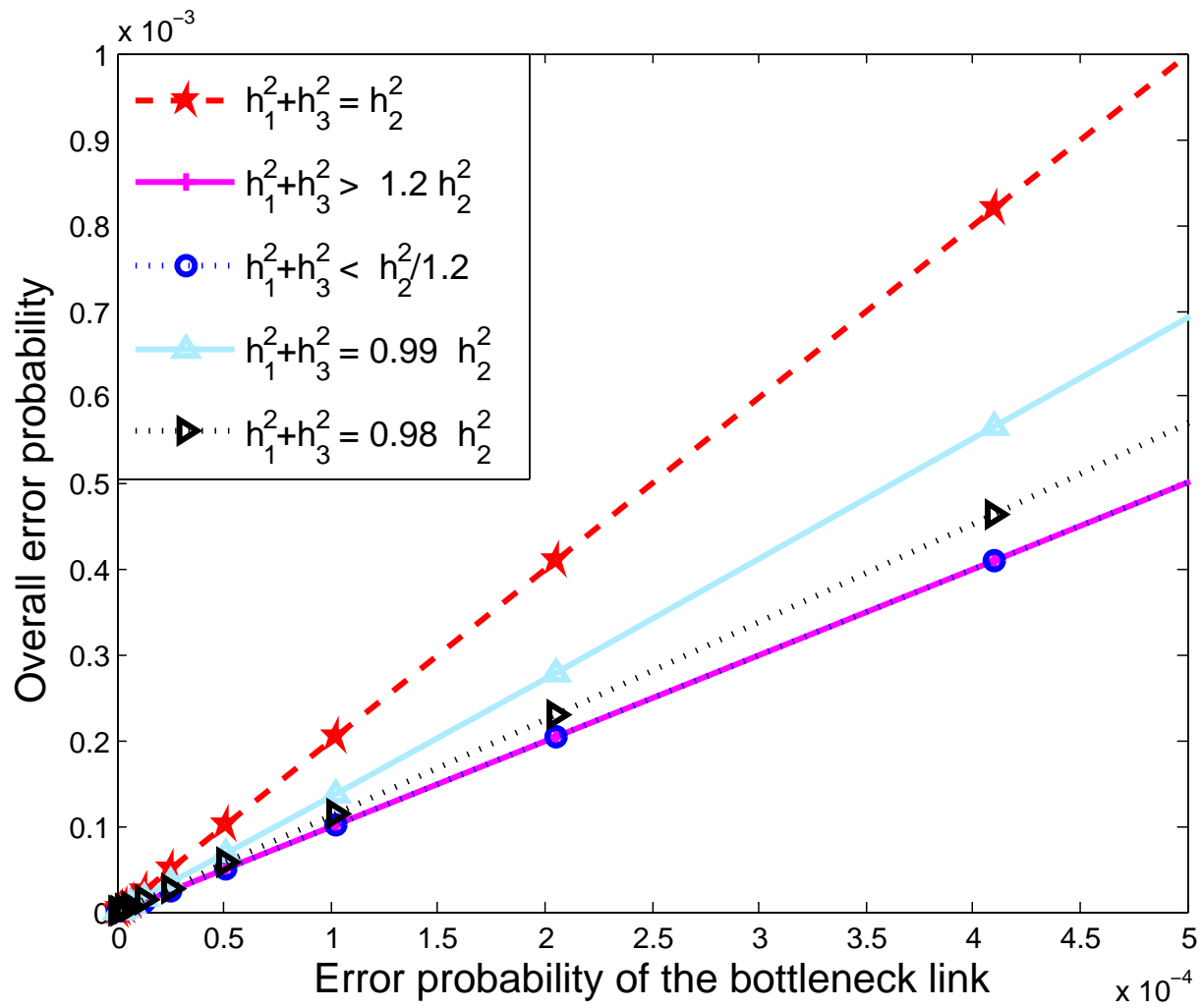


Fig. 2. The relationship between overall error probability and error probability of the bottleneck link.

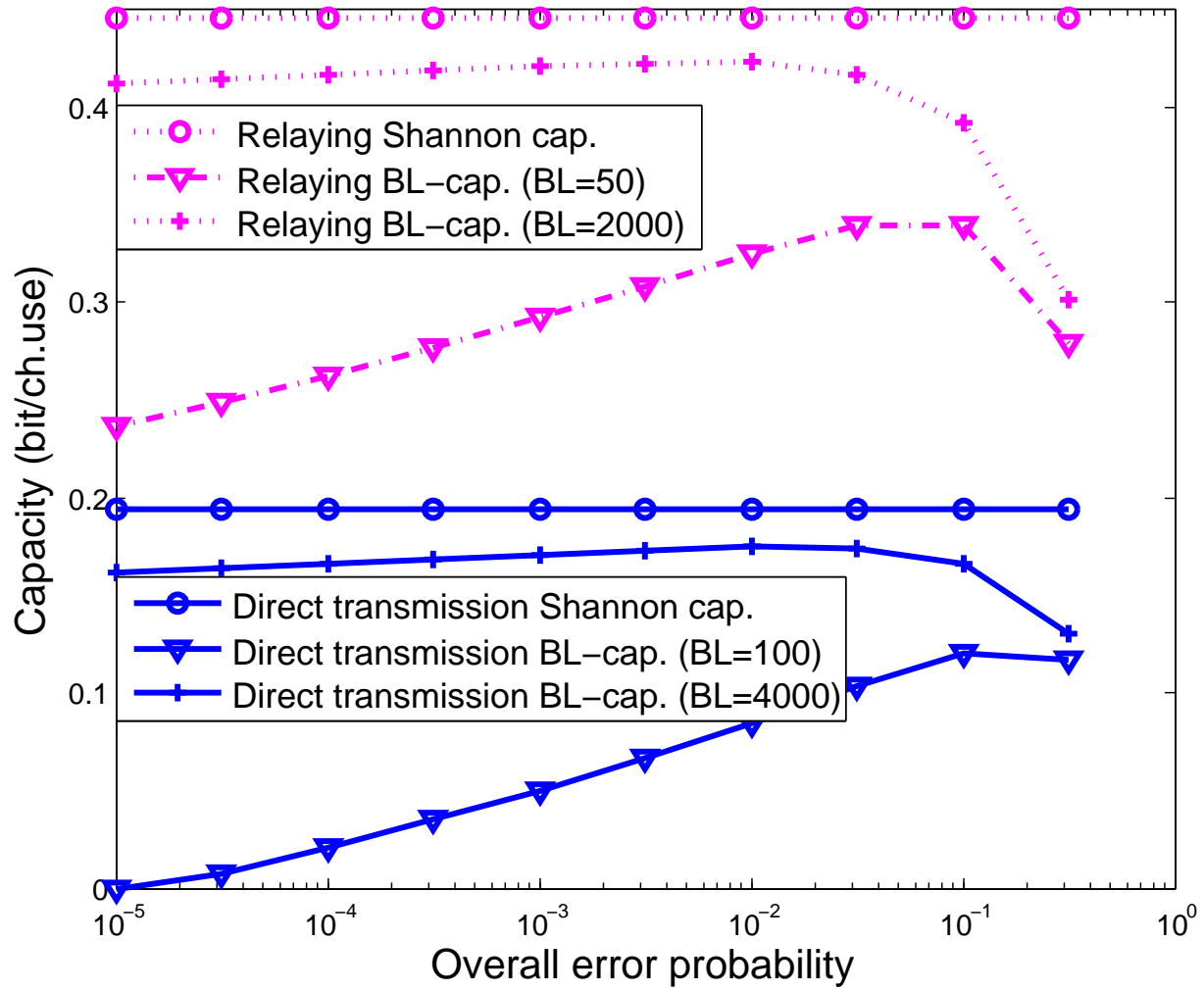


Fig. 3. BL-capacities under static channels with $\gamma_1 = -5$ dB, $\gamma_2 = \gamma_3 = 4$ dB and $\gamma_{\text{MRC}} = 4.5$ dB.

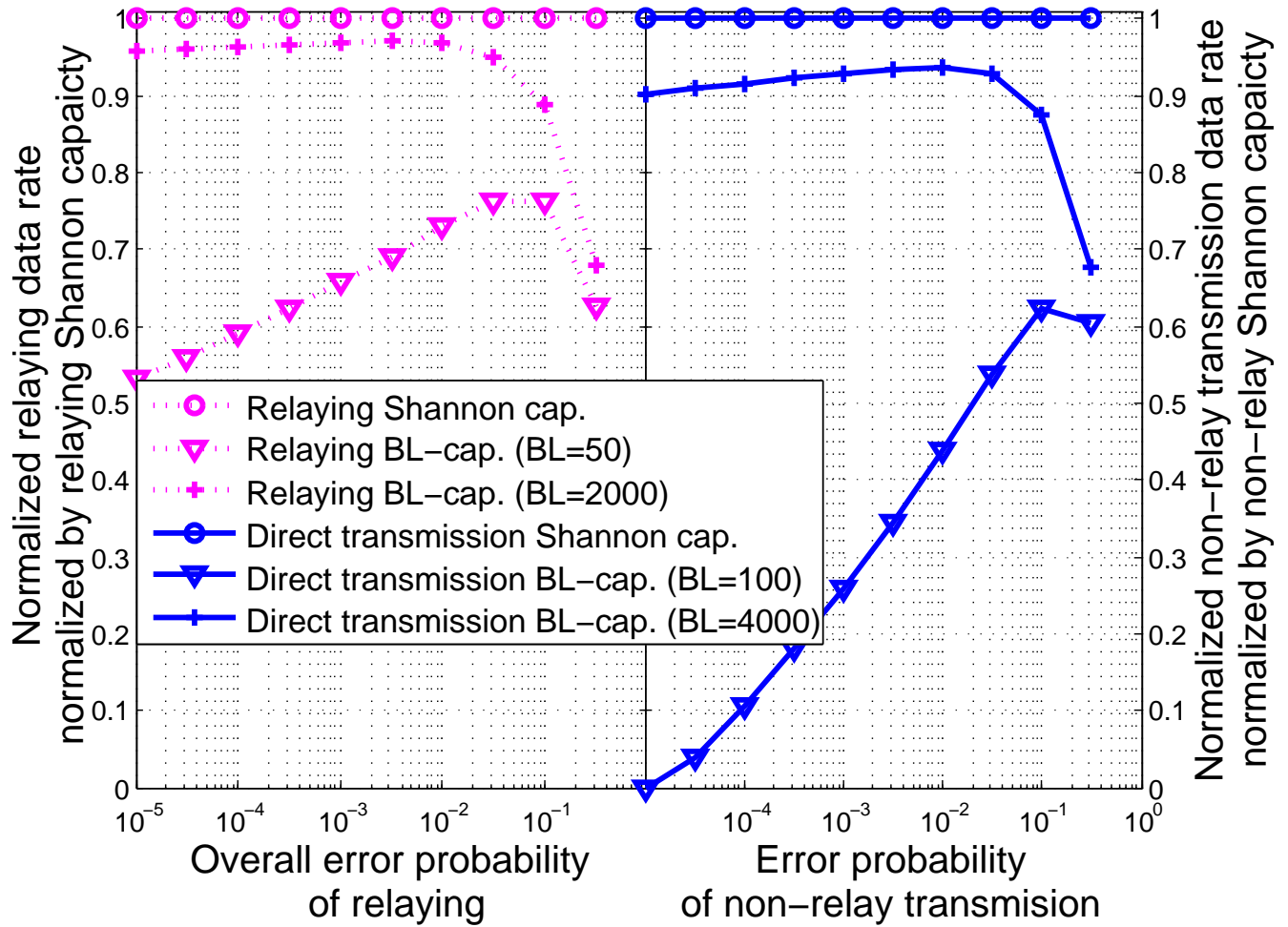


Fig. 4. The normalized data rates under static channels.